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# Continuous Dependence on a Parameter of the Solutions of Impulsive Differential Equations in a Banach Space

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We prove that the solutions of an impulsive differential equation depend continuously on a small parameter under the assumption that the right-hand side of the equation and the impulse operators satisfy conditions of Lipschitz type.

## 1. INTRODUCTION

The dependence on a parameter of the solutions of impulsive differential equations in a Banach space is investigated. A theorem is proved which generalizes some results of Daleckii and Krein (1974) even for the case of a differential equation without impulse effect.

## 2. STATEMENT OF THE PROBLEM

Let $X$ be an arbitrary Banach space with norm $\|\cdot\|_X = \|\cdot\|$. Consider the impulsive differential equation

$$\frac{dx}{d\tau} = f(\tau, x, \varepsilon) \qquad (\tau \neq t_n) \tag{1}$$

$$\Delta x|_{\tau = t_n} = I_n(x(t_n), \varepsilon) \qquad (n = 1, \ldots, p) \tag{2}$$

$0 \leqslant \tau \leqslant T$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$ ($T$ and $\varepsilon_0$ are constants), where $x(t) \in X$ ($0 \leqslant t \leqslant T$), $f(t, x, \varepsilon) \in X$ ($0 \leqslant t \leqslant T$, $x \in X$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$), $t_n < t_{n+1}$ ($n = 1, \ldots, p-1$; $p$ is

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the number of the points of impulse effect in the interval $[0, T]$), and $I_n(x, \varepsilon) \in X$ $(x \in X,\ 0 \leqslant \varepsilon \leqslant \varepsilon_0)$.

We shall say that conditions (H) are met if the following conditions hold:

H1. $\|f(\tau, x_2, \varepsilon) - f(\tau, x_1, \varepsilon)\| \leqslant c(\tau, \varepsilon)\, \|x_2 - x_1\|$ $(0 \leqslant \tau \leqslant T;\ x_1, x_2 \in X;\ 0 \leqslant \varepsilon \leqslant \varepsilon_0)$

H2. $\|I_k(x_2, \varepsilon) - I_k(x_1, \varepsilon)\| \leqslant d_k(\varepsilon)\, \|x_2 - x_1\|$ $(k = 1, \ldots, p;\ x_1, x_2 \in X;\ 0 \leqslant \varepsilon \leqslant \varepsilon_0)$

H3. $\lim_{\varepsilon \to 0} \int_0^\tau f(\sigma, x, \varepsilon)\, d\sigma = \int_0^\tau f(\sigma, x, 0)\, d\sigma$ $(0 \leqslant \tau \leqslant T,\ x \in X)$

H4. $\lim_{\varepsilon \to 0} I_k(x, \varepsilon) = I_k(x, 0)$ $(k = 1, \ldots, p;\ x \in X)$

Let $Y$ be an arbitrary Banach space.

By $\tilde{C}([0, T], Y)$ we denote the set of all functions $x$: $[0, T] \to Y$ which are continuous for $t \neq t_n$ and have discontinuities of the first kind at the points $t_n$, where they are continuous from the left. With respect to the norm $\|x\|_{\tilde{C}} = \sup_{0 \leqslant t \leqslant T} \|x(t)\|$, $\tilde{C}([0, T], Y)$ is a Banach space.

*Lemma 1.* Let the following conditions hold:

1. Conditions H1 and H3 are met.
2. $\int_0^T c(\sigma, \varepsilon)\, d\sigma \leqslant M$ $(0 \leqslant \varepsilon \leqslant \varepsilon_0)$.
3. $x \in \tilde{C}([0, T], X)$.

Then in $\tilde{C}([0, T], X)$ the following equality is valid:

$$\lim_{\varepsilon \to 0} \int_0^\tau f(\sigma, x(\sigma), \varepsilon)\, d\sigma = \int_0^\tau f(\sigma, x(\sigma), 0)\, d\sigma \qquad (0 \leqslant \tau \leqslant T) \tag{3}$$

*Proof.* From H3 it follows that for $\tau_1, \tau_2 \in [0, T]$ and $x \in X$ the equality

$$\lim_{\varepsilon \to 0} \int_{\tau_1}^{\tau_2} f(\sigma, x, \varepsilon)\, d\sigma = \int_{\tau_1}^{\tau_2} f(\sigma, x, 0)\, d\sigma$$

is valid; hence for arbitrarily chosen $0 \leqslant \tau_1 < \cdots < \tau_{n-1} < \tau_n = T$, $x_k \in X$ $(k = 1, \ldots, n)$ we have

$$\lim_{\varepsilon \to 0} \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} f(\sigma, x_k, \varepsilon)\, d\sigma = \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} f(\sigma, x_k, 0)\, d\sigma \tag{4}$$

Consider the step function

$$\tilde{x}(\tau) = x_k \qquad (\tau_k \leqslant \tau \leqslant \tau_{k+1};\ k = 1, \ldots, n-1)$$

by means of which equality (4) takes the form

$$\lim_{\varepsilon \to 0} \int_0^{\tau} f(\sigma, \tilde{x}(\sigma), \varepsilon)\, d\sigma = \int_0^{\tau} f(\sigma, \tilde{x}(\sigma), 0)\, d\sigma \tag{5}$$

Let $\{\tilde{x}_m(\sigma)\}_{m=1}^{\infty}$ be a sequence of step functions tending uniformly on $[0, T]$ to $x(\tau)$. From H1 and condition 2 of Lemma 1 we obtain the inequalities

$$\begin{aligned}
&\left\| \int_0^{\tau} f(\sigma, x(\sigma), \varepsilon)\, d\sigma - \int_0^{\tau} f(\sigma, x(\sigma), 0)\, d\sigma \right\| \\
&\quad \leqslant \int_0^{\tau} \|f(\sigma, x(\sigma), \varepsilon) - f(\sigma, \tilde{x}_m(\sigma), \varepsilon)\|\, d\sigma \\
&\qquad + \left\| \int_0^{\tau} [f(\sigma, \tilde{x}_m(\sigma), \varepsilon) - f(\sigma, \tilde{x}_m(\sigma), 0)]\, d\sigma \right\| \\
&\qquad + \int_0^{\tau} \|f(\sigma, \tilde{x}_m(\sigma), 0) - f(\sigma, x(\sigma), 0)\|\, d\sigma \\
&\quad \leqslant \sup_{0 \leqslant \sigma \leqslant T} \|x(\sigma) - \tilde{x}_m(\sigma)\| \int_0^{\tau} c(\sigma, \varepsilon)\, d\sigma + \left\| \int_0^{\tau} [f(\sigma, \tilde{x}_m(\sigma), \varepsilon) \right. \\
&\qquad \left. - f(\sigma, \tilde{x}_m(\sigma), 0)]\, d\sigma \right\| + \sup_{0 \leqslant \sigma \leqslant T} \|x(\sigma) - \tilde{x}_m(\sigma)\| \int_0^{T} c(\sigma, 0)\, d\sigma \\
&\quad \leqslant 2M \sup_{0 \leqslant \sigma \leqslant T} \|x(\sigma) - \tilde{x}_m(\sigma)\| \\
&\qquad + \left\| \int_0^{\tau} [f(\sigma, \tilde{x}_m(\sigma), \varepsilon) - f(\sigma, \tilde{x}_m(\sigma), 0)]\, d\sigma \right\|
\end{aligned} \tag{6}$$

The proof of Lemma 1 follows from inequalities (6). ■

## 3. MAIN RESULTS

*Theorem 1.* Let the following conditions hold:

1. The function $f(\tau, x, \varepsilon)$ $(0 \leqslant \tau \leqslant T,\ x \in X,\ 0 \leqslant \varepsilon \leqslant \varepsilon_0)$, is uniformly bounded on each ball $B \subset X$ and is piecewise continuous with respect to $\tau$.
2. Conditions (H) hold.
3. The impulsive equation (1), (2) has for $\varepsilon = 0$ a solution $x(\sigma, 0)$ $(0 \leqslant \sigma \leqslant T)$.
4. $\int_0^T c(\sigma, \varepsilon)\, d\sigma \leqslant M$, $\prod_{k=1}^{p} [1 + d_k(\varepsilon)] \leqslant M$ ($M$ is a constant).

Then for any $\varepsilon \in [0, \varepsilon_0]$ equation (1), (2) has a unique solution $x(\sigma, \varepsilon) \in \tilde{C}([0, T], X)$ for which

$$x(0, \varepsilon) = x(0, 0) = x_0$$

Moreover, for any $\eta > 0$ there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for $0 < \varepsilon < \varepsilon_1$ the following estimate is valid:

$$\|x(\tau, \varepsilon) - x(\tau, 0)\| < \eta \qquad (0 \leqslant \tau \leqslant T)$$

*Proof.* It is immediately verified that the impulsive equation (1), (2) is equivalent to the nonlinear equation

$$x(\tau, \varepsilon) = x_0 + \int_0^\tau f(s, x(s, \varepsilon), \varepsilon)\, ds + \sum_{0 \leqslant t_k < \tau} I_k(x(t_k, \varepsilon), \varepsilon) \tag{7}$$

That is why in order to prove the theorem it suffices to show that equation (7) for sufficiently small $\varepsilon$ has a solution $x(\tau, \varepsilon)$ which, as $\varepsilon \to 0$, tends with respect to the norm of the space $\tilde{C}([0, T], X)$ to the solution $x(\sigma, 0)$ of this equation for $\varepsilon = 0$.

Consider the auxiliary operator $Q$ acting in the space $\tilde{C}([0, T], \mathbb{R})$ and defined by the formula

$$Q(\varepsilon)\, z(\tau) = \int_0^\tau c(s, \varepsilon)\, z(s)\, ds + \sum_{0 < t_k < \tau} d_k(\varepsilon)\, z(t_k) \tag{8}$$

For arbitrary $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and for any function $f(\sigma) \in \tilde{C}([0, T], \mathbb{R})$ the equation

$$\lambda z(\tau) - Q(\varepsilon)\, z(\tau) = f(\tau) \tag{9}$$

has a unique solution $z(\tau) \in \tilde{C}([0, T], \mathbb{R})$ which can be constructed by Volterra's method of successive approximations consecutively on each of the intervals $[t_j, t_{j+1}]$ $(j = 1, \ldots, p-1)$.

The solvability of equation (9) for $\lambda \neq 0$ in $\tilde{C}([0, T], \mathbb{R})$ means that the spectral radius of the operator $Q(\varepsilon)$ in $\tilde{C}([0, T], \mathbb{R})$ equals zero. That is why (Krasnosel'skii *et al.*, 1972) in the space $\tilde{C}([0, T], \mathbb{R})$ there exists a norm equivalent to the initial one. Moreover, the inequality

$$\|Q(\varepsilon)\|_* \leqslant q \tag{10}$$

is valid, where $q \in (0, 1)$ is a given number and $\|\cdot\|_*$ is the operator norm corresponding to the new norm in $\tilde{C}([0, T], \mathbb{R})$.

Set

$$A(\varepsilon, x)(\tau) = x_0 + \int_0^\tau f(s, x(s), \varepsilon)\, ds + \sum_{0 \leqslant t_k < \tau} I_k(x(t_k), \varepsilon)$$

Obviously we have

$$\begin{aligned}
&\|A(\varepsilon, x_1)(\tau) - A(\varepsilon, x_2)(\tau)\| \\
&\quad \leqslant \int_0^\tau \|f(s, x_1(s), \varepsilon) - f(s, x_2(s), \varepsilon)\| \, ds \\
&\qquad + \sum_{0 < t_k < \tau} \|I_k(x_1(t_k), \varepsilon) - I_k(x_2(t_k), \varepsilon)\| \\
&\quad \leqslant \int_0^\tau c(s, \varepsilon) \, \|x_1(s) - x_2(s)\| \, ds \\
&\qquad + \sum_{0 < t_k < \tau} d_k(\varepsilon) \, \|x_1(t_k) - x_2(t_k)\| \\
&\quad = Q(\varepsilon) \, \|x_1(\tau) - x_2(\tau)\|
\end{aligned} \tag{11}$$

From (10) and (11) it follows that

$$\| \, \| A(\varepsilon, x_1)(\tau) - A(\varepsilon, x_2)(\tau) \| \, \|_* \leqslant q \, \| \, \| x_1(\tau) - x_2(\tau) \| \, \|_* \tag{12}$$

From (12) it follows that with respect to the norm $\|\cdot\|_{**} = \| \, \|\cdot\|X\|_*$ of the space $\tilde{C}([0, T], X)$ the operator $A(\varepsilon, x)$ satisfies the Lipschitz condition with a constant $q < 1$.

From Lemma 1 it follows that

$$\lim_{\varepsilon \to 0} \|x(\tau, 0) - A(\varepsilon, x(\tau, 0))\| = 0$$

hence the operator $A(\varepsilon, x)$ satisfies in the space $\tilde{C}([0, T], X)$ with norm $\|\cdot\|_{**}$ the conditions of the Banach–Caccioppoli contracting mapping principle. Hence for small values of $\varepsilon$ the operator $A(\varepsilon, x)$ has in this space a unique fixed point which, as $\varepsilon \to 0$ in the norm $\|\cdot\|_{**}$, and therefore in the initial norm as well, tends to $x(\tau, 0)$.

Theorem 1 is proved. ■

As an application of Theorem 1, we shall consider the particular case when the impulsive equation has the form

$$\frac{dx}{d\tau} = f\left(\frac{\tau}{\varepsilon}, x\right) \qquad (\tau \neq t_n; \; n = 1, \ldots, p) \tag{13}$$

$$x(t_n^+) - x(t_n) = I_n(x(t_n), \varepsilon) \qquad (n = 1, \ldots, p) \tag{14}$$

We shall say that conditions (A) are met if the following conditions hold:

A1. The function $f(\tau, x)$ $(0 \leqslant \tau \leqslant T, \; x \in X)$ is uniformly bounded on each ball $B \subset X$ and is piecewise continuous with respect to $\tau$.

A2. $\|f(\tau, x_2) - f(\tau, x_1)\| \leqslant c \, \|x_2 - x_1\|$ $(0 \leqslant \tau \leqslant T)$

A3. For any fixed $x \in X$ there exists the temporal mean

$$f_0(x) = \lim_{\omega \to \infty} \frac{1}{\omega} \int_0^{\omega} f(t, x)\, dt$$

A4. There exist the limits

$$I_k(x) = \lim_{\varepsilon \to 0} I_k(x, \varepsilon) \qquad (k = 1, \dots, p)$$

Consider the impulse equation

$$\frac{dx}{d\tau} = f_0(x) \qquad (\tau \neq t_n) \tag{15}$$

$$x(t_n^+) - x(t_n) = I_n(x(t_n)) \qquad (n = 1, \dots, p) \tag{16}$$

*Corollary 1.* Let conditions (A) hold and let equation (15), (16) have a solution $x_0(t)$ which is defined on $[0, T]$.

Then for any $\eta > 0$ equation (13), (14) for sufficiently small $\varepsilon$ has a solution $x(t, \varepsilon)$ which is defined on $[0, T]$ and for which the following inequality is valid:

$$\|x(t, \varepsilon) - x_0(t\varepsilon)\| < \eta$$

*Remark 1.* The assertion of Corollary 1 can be considered as an analog of one of the fundamental theorems of the Bogolyubov–Krylov averaging principle (Daleckii and Krein, 1974).

*Remark 2.* Theorem 1 and Corollary 1 can be easily reformulated for the case when the functions $f$ and $I_k$ are defined only on some closed ball of the space $X$.

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